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## Lift-contractions

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## ABSTRACT

We introduce and study a partial order on graphs—lift-contractions. A graph  $H$  is a lift-contraction of a graph  $G$  if  $H$  can be obtained from  $G$  by a sequence of edge lifts and edge contractions. We give sufficient conditions for a connected graph to contain every  $n$ -vertex graph as a lift-contraction and describe the structure of graphs with an excluded lift-contraction.

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## 1. Introduction

All graphs in this paper are undirected, loopless, and without multiple edges (unless mentioned otherwise).  $V(G)$  and  $E(G)$  denote the vertex and edge set of a graph  $G$ , respectively. The *degree* of a vertex  $v \in V(G)$  is the number of edges incident with it.  $K_n$  is the complete graph on  $n$  vertices. Given an edge  $e$  of a graph  $G$ , the result of the *contraction* of  $e$  in  $G$  is the graph obtained by removing  $e$  from  $G$  and then identifying its endpoints to a single vertex  $v_e$ . For notions and notations not defined here, we refer the reader to the monograph [5].

Given two edges  $e_1 = \{x, x_1\}$  and  $e_2 = \{x, x_2\}$  of  $G$ , incident with the same vertex  $x$ , and such that  $x_1 \neq x_2$ , we define the *lift* of  $e_1$  and  $e_2$  in  $G$  as the graph obtained by removing  $e_1$  and  $e_2$  from  $G$  and then adding the edge  $\{x_1, x_2\}$ . If a contraction or edge lift creates multiple edges, we reduce their multiplicity to one and keep the graph simple.

*Partial orders.* The study of partial orders on graphs is one of the basic research avenues in graph theory. One of the most comprehensive studies of partial orders is the theory of Graph Minors by Robertson and Seymour [11] (see also the last chapter of [5]). A graph  $H$  is a *minor* of another graph  $G$  ( $H \leq_m G$ )

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if  $H$  can be obtained from  $G$  by a sequence of vertex deletions, edge removals, and edge contractions. Some more restricted graph containment relations than graph minors, like contractions [3] or induced minors [9] have also been studied.

Graph immersions form another partial order that has been considered in the literature [4]. A graph  $H$  is an *immersion* of  $G$  if  $H$  can be obtained from  $G$  by a sequence of vertex deletions, edge removals, and edge lifts. The last operation was introduced by Lovász under the name of *splitting* as a reduction method to maintain edge connectivity [8].

In this paper, we introduce and study lift-contractions. We say that a graph  $H$  is a *lift-contraction* of a graph  $G$  if  $H$  can be obtained from  $G$  by a sequence of edge lifts and edge contractions. We also define lift-minors. We say that a graph  $H$  is a *lift-minor* of a graph  $G$  if  $H$  can be obtained from  $G$  by a sequence of vertex and edge deletions, edge lifts and contractions.

Being a lift-contraction (lift-minor) is a partial relation between graphs and we denote it by  $H \leq_{lc} G$  ( $H \leq_{lm} G$ ). If a graph  $H$  can be obtained from  $G$  by a sequence of contractions, we say that  $H$  is a contraction of  $G$  and we denote this by  $H \leq_c G$ . Clearly,  $H \leq_c G \Rightarrow H \leq_{lc} G \Rightarrow H \leq_{lm} G$  and  $H \leq_m G \Rightarrow H \leq_{lm} G$ .

**Forcing complete graphs.** When studying a partial order  $\leq$  on graphs, it is interesting to know under what conditions on  $G$ , for a fixed graph  $H$ ,  $H \leq G$ . Kostochka [7] and Thomason [13] independently proved that if the average degree of  $G$  is at least  $cn\sqrt{\log n}$ , then  $G$  contains  $K_n$  as a minor (for some constant  $c > 0$ ). Bollobás [2] showed that if the average degree of  $G$  is at least  $cn^2$ , then  $G$  contains  $K_n$  as a topological minor<sup>1</sup> (for some constant  $c > 0$ ). Recently, DeVos, Dvořák, Fox, McDonald, Mohar and Scheide [4] proved that if the minimum degree of  $G$  is at least  $200n$ , then  $G$  contains  $K_n$  as an immersion. For all these three partial orders, containing  $K_n$  implies containing any  $n$ -vertex graph.

In this paper, we identify three conditions on a connected graph  $G$  that force any  $n$ -vertex graph as a lift-contraction of  $G$ .

**Theorem 1.1.** *There exists a constant  $c$  such that every connected graph  $G$  of treewidth at least  $c \cdot n^4$  contains every  $n$ -vertex graph as a lift-contraction.*

**Theorem 1.2.** *There exists a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that every 2-connected graph of pathwidth at least  $f(n)$  contains every  $n$ -vertex graph as a lift-contraction.*

**Theorem 1.3.** *There exists a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that every connected graph with at least  $f(n)$  vertices and minimum degree at least 3 contains every  $n$ -vertex graph as a lift-contraction.*

We note that none of the three conditions above is alone enough to force all  $n$ -vertex graphs as a lift or as a contraction. In order to see this, consider a complete graph  $K$  with an arbitrarily large number of vertices. Because an edge lift does not change the number of vertices, we cannot obtain a graph with fewer vertices than  $K$  by taking edge lifts only. Because contracting an edge in  $K$  yields a new complete graph, we cannot obtain any non-complete graph by performing edge contractions only.

**Structural theorem.** Another point of focus, when studying partial orders on graphs, is to understand the structure of nontrivial ideals in this order. The best known example is the structural theorem on graphs with an excluded minor by Robertson and Seymour [11]. Recently, a structural description of graphs with an excluded topological minor was discovered by Grohe and Marx [6] and with an excluded immersion by Wollan [14].

Here we obtain, as a consequence of Theorem 1.3, a structural description of graphs with a forbidden lift-contraction. Informally, for a fixed graph  $H$ , any graph  $G$  that does not contain  $H$  as a lift-contraction contains a set of vertices  $R$  whose size depends only on the excluded graph  $H$  such that every connected component of  $G[V \setminus R]$  is of treewidth at most 2 and has at most two neighbors in  $R$ . A simple corollary of our structural result is that graphs with an excluded lift-contraction are of bounded treewidth and thus of bounded chromatic number.

<sup>1</sup>  $H$  is a topological minor of  $G$ , when some subdivision of  $H$  is a subgraph of  $G$ .

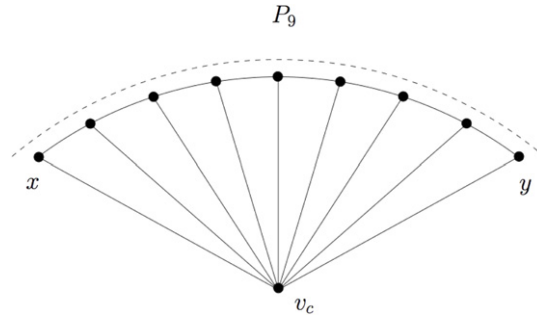


Fig. 1. The graph  $F_9$ .

*Paper structure.* We start with some preliminary results in Section 2 which also includes the proofs of Theorems 1.1 and 1.2. The proof of Theorem 1.3 is presented in Section 3. In Section 4 we describe the structure of graphs with an excluded lift-minor.

## 2. Preliminary results

We prove below auxiliary results that will be useful later in the next sections, give some definitions and prove Theorems 1.1 and 1.2.

**Lemma 2.1.** *For every  $n$ -vertex graph  $H$ ,  $H \leq_{lc} K_{2n}$ .*

**Proof.** We prove that every  $n$ -vertex graph  $H$  is a lift-contraction of  $K_{2n}$ . Let  $H^+ = K_2 \times H$ . First we prove that  $H^+$  is a lift of  $K_{2n}$ . Let  $V(H) = \{v_1, \dots, v_n\}$  and  $V(H^+) = \{v'_1, \dots, v'_n, v''_1, \dots, v''_n\}$ . Let us assume that  $V(K_{2n}) = V(H^+)$  and observe that  $H^+$  is a spanning subgraph of  $K_{2n}$ . Let  $R$  be the set of non-edges of  $H$ , i.e.,  $R = \{\{u, v\} \mid u, v \in V(H), u \neq v\} \setminus E(H)$ . Notice that each  $\{v_i, v_j\} \in R$  corresponds to the vertices  $v'_i, v'_j, v''_i, v''_j \in V(H^+)$  such that the edges  $\{v'_i, v'_j\}, \{v''_i, v''_j\}, \{v'_i, v''_j\}, \{v''_i, v'_j\}$  are present in  $K_{2n}$  but not in  $H^+$ . We use edge lifts to remove those edges. For every  $\{v_i, v_j\} \in R$ , we lift the pairs of edges  $\{v'_i, v'_j\}, \{v''_j, v'_i\}$  and  $\{v''_i, v'_j\}, \{v'_j, v''_i\}$ . The result is  $H^+$ . Now we contract edges  $\{v'_i, v''_i\}$  for all  $i = 1, \dots, n$  and obtain  $H$  as claimed.  $\square$

The following observation can be easily proved by induction on  $r$ .

**Observation 2.2.** *For every  $r \geq 2$ , the complete  $r$ -partite graph, where each of its parts has  $r - 1$  vertices, has a perfect matching  $M$  such that for every two of its parts there is exactly one edge in  $M$  intersecting both of them.*

For an integer  $k > 1$ , the  $k$ -fan is the graph obtained from the path  $P_k$  on  $k$  vertices by adding a dominating vertex  $v_c$ . We denote the  $k$ -fan by  $F_k$  and say that  $P_k$  is its *spine* and  $v_c$  is its *center* (see Fig. 1). The *extreme vertices* of a  $k$ -fan are the endpoints of the path (i.e., the vertices  $x$  and  $y$  in Fig. 1).

**Lemma 2.3.** *For any connected graph  $G$  and  $n \geq 2$ , if  $F_{n(n-1)} \leq_{lm} G$ , then  $K_n \leq_{lc} G$ .*

**Proof.** If  $F_{n(n-1)} \leq_{lm} G$ , then it is possible to obtain  $F_{n(n-1)}$  from  $G$  by a sequence of vertex deletions, edge removals, edge contractions and edge lifts. We modify this sequence as follows:

- a removal of an edge  $e$  such that  $e$  is a bridge in the already constructed graph is replaced by the contraction of  $e$ , all other edge removals are deleted from the sequence;
- a removal of a vertex  $v$  is replaced by the contraction of an edge incident with  $v$ ;
- a lift operation for edges  $\{u, v\}, \{v, w\}$  such that  $v$  has degree 2 in the already constructed graph is replaced by the contraction of  $\{u, v\}$ .

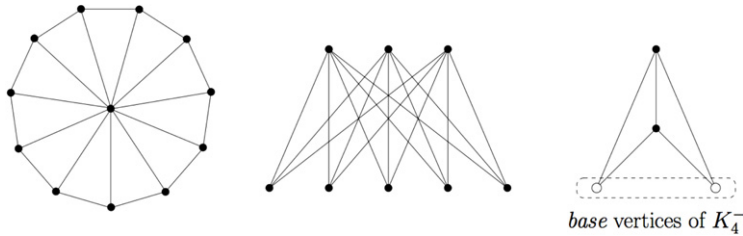


Fig. 2. The graphs  $W_{11}$ ,  $K_{3,5}$ , and  $K_4^-$ .

By the resulting sequence of contractions and edge lifts, we obtain a graph  $G' \leq_{lc} G$  such that  $G'$  contains  $F_{n(n-1)}$  as a spanning subgraph. Let the spine of this  $n(n-1)$ -fan in  $G'$  be a path  $P$  with  $V_P = \{v_1^1, \dots, v_{n-1}^1, v_1^2, \dots, v_{n-1}^2, \dots, v_1^n, \dots, v_{n-1}^n\}$ . Let  $J$  be the complete  $n$ -partite graph with partition classes  $\{v_1^1, \dots, v_{n-1}^1\}, \dots, \{v_1^n, \dots, v_{n-1}^n\}$ , and let  $M$  be a perfect matching of  $J$  as in Observation 2.2. We choose an arbitrary edge  $\{v_s^t, v_{s'}^{t'}\} \in M$ . For each edge  $\{v_i^j, v_{i'}^{j'}\} \in M$ , where  $\{v_s^t, v_{s'}^{t'}\} \neq \{v_i^j, v_{i'}^{j'}\}$ , we lift the pair of edges  $\{v_i^j, v_c\}$  and  $\{v_{i'}^{j'}, v_c\}$  in  $G'$ . Then we contract  $\{v_s^t, v_{s'}^{t'}\}$ . In the resulting graph, we contract, for each  $i \in \{1, \dots, n\}$ , all the edges in  $\{\{v_i^j, v_{i+1}^j\} \mid j \in \{1, \dots, n-1\}\}$  to a single vertex  $u^i$ . Observe that the resulting graph is a complete graph with the vertex set  $\{u^1, \dots, u^n\}$ . Hence,  $K_n \leq_{lc} G' \leq_{lc} G$  as claimed.  $\square$

A *tree decomposition* of a graph  $G$  is a pair  $(\mathcal{X}, T)$  where  $T$  is a tree and  $\mathcal{X} = \{X_i \mid i \in V(T)\}$  is a collection of subsets of  $V(G)$  (called bags) such that:

1.  $\bigcup_{i \in V(T)} X_i = V(G)$ ;
2. for each edge  $\{x, y\} \in E(G)$ ,  $\{x, y\} \subseteq X_i$  for some  $i \in V(T)$ , and
3. for each  $x \in V(G)$  the set  $\{i \mid x \in X_i\}$  induces a connected subtree of  $T$ .

The *adhesion* of a tree decomposition  $(\{X_i \mid i \in V(T)\}, T)$  is  $\max\{|X_i \cap X_j| \mid i, j \in V(T), i \neq j\}$  and its width is  $\max\{|X_i| - 1 \mid i \in V(T)\}$ . The *treewidth* of a graph  $G$  is the minimum width over all tree decompositions of  $G$ . A *path decomposition* of a tree decomposition where the tree  $T$  is a path. The *pathwidth* of a graph  $G$  is the minimum width of a path decomposition of it.

**Proof of Theorem 1.1.** From Lemmas 2.1 and 2.3,  $G$  does not contain  $F_{2n^2-2n}$  as a minor; otherwise we are done. A graph with no  $K_2 \times C_k$  minor, where  $C_k$  is a cycle on  $k$  vertices, has treewidth at most  $60k^2 - 120k + 63$  [1]. As  $F_k$  is a minor of  $K_2 \times C_k$ , the same bound holds for graphs with no  $F_k$  minor. The result follows by taking  $k = 2n^2 - 2n$ .  $\square$

**Proof of Theorem 1.2.** According to a result mentioned in [12], for any pair of graphs  $G$  and  $H$  such that  $G$  is an outerplanar graph and  $H$  has a vertex whose removal leaves a tree, there is a constant  $c_{G,H}$  such that every 2-connected graph of pathwidth at least  $c_{G,H}$  contains  $G$  or  $H$  as a minor. By taking both  $G$  and  $H$  to be a  $k$ -fan, we conclude that there is a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that every 2-connected graph of pathwidth at least  $f(k)$  contains  $F_k$  as a lift-minor. Then Lemma 2.3 yields the result.  $\square$

### 3. Proof of Theorem 1.3

Let  $W_k$  be the graph obtained from  $F_k$  by adding an edge between its extreme vertices (assuming that  $k \geq 3$ ). Let  $K_{3,k}$  be the complete bipartite graph whose parts have exactly 3 and  $k$  vertices. We denote by  $K_4^-$  the graph obtained from  $K_4$  by removing an edge, and we call the vertices of degree 2 in it *base vertices*. Examples of these graphs are shown in Fig. 2. We let  $\Gamma_r = K_2 \times P_r$ . We denote by  $M_r$  the graph obtained if we take  $r$  copies of  $K_4$ , pick a vertex in each of them, and then identify all chosen vertices to a single vertex. We denote by  $N_r$  the graph obtained as follows. We take  $r$  copies of  $K_4^-$ . In each copy we choose an arbitrary base vertex and call it a *left* base vertex, and say that another base vertex is *right*. Then we identify all left vertices and all right vertices. Finally, we denote by  $L_r$  the graph

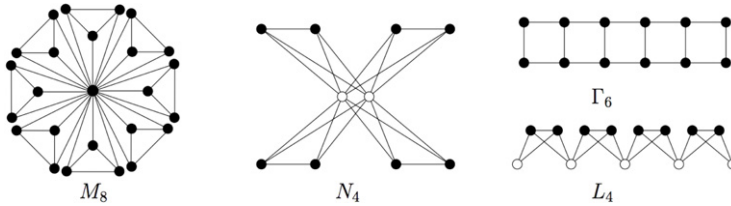


Fig. 3. The graphs  $M_8$ ,  $N_4$ ,  $\Gamma_6$ , and  $L_4$ .

obtained if we take  $r$  copies of  $K_4^-$ , pick a left and right base vertices in each copy, and then identify the right base vertex of the  $(i-1)$ -th copy and the left base vertex of the  $i$ -th copy for  $i \in \{2, \dots, r\}$ . See Fig. 3 for examples.

We need the following lemmas.

**Lemma 3.1.** *For each  $k \geq 1$ , it holds that  $F_k \leq_{\text{lm}} W_k$ ,  $F_k \leq_{\text{lm}} K_{3,k}$ ,  $F_k \leq_{\text{lm}} M_k$ ,  $F_k \leq_{\text{lm}} N_k$ ,  $F_k \leq_{\text{lm}} \Gamma_k$ , and  $F_k \leq_{\text{lm}} L_k$ .*

**Proof.** Clearly,  $F_k \leq_{\text{lm}} W_k$ . Because  $\Gamma_k$  consists of two paths on  $k$  vertices joined by a matching, it is straightforward to see that we obtain  $F_k$  by contracting all the edges of one path.

For  $K_{3,k}$ , denote by  $u_1, u_2, u_3$  and  $v_1, \dots, v_k$  the vertices of the respective partition sets. For  $i \in \{1, \lfloor k/2 \rfloor\}$ , we lift the edges  $\{v_{2i-1}, u_1\}$ ,  $\{u_1, v_{2i}\}$ , and for  $i \in \{1, \lfloor k/2 \rfloor - 1\}$ ,  $\{v_{2i}, u_2\}$ ,  $\{u_2, v_{2i+1}\}$  are lifted. Now  $F_k$  with the center  $u_3$  is a subgraph of the obtained graph.

Recall that  $M_k$  is obtained from  $k$  copies of  $K_4$  by identifying vertices chosen in each copy. Let  $x_i, y_i, z_i$  and  $v$  be the vertices of the  $i$ -th copy ( $v$  is a common vertex) for  $i \in \{1, \dots, k\}$ . We obtain  $F_k$  as follows: for  $i \in \{1, \dots, k-1\}$ , we lift  $\{x_i, v\}$ ,  $\{v, y_{i+1}\}$ , and then  $x_i, y_i, z_i$  are contracted to a single vertex for all  $i \in \{1, \dots, k\}$ .

Consider now  $N_k$  obtained from  $k$  copies of  $K_4^-$ . Let  $x_i, y_i, v_l, v_r$  be the vertices of the  $i$ -th copy where  $v_l, v_r$  are the common base vertices for  $i \in \{1, \dots, k\}$ . For  $i \in \{1, \dots, k-1\}$ , we lift  $\{x_i, v_l\}$ ,  $\{v_l, y_{i+1}\}$  and observe that  $F_k$  is a subgraph of the obtained graph.

Finally, assume that  $L_k$  consists of  $k$  copies of  $K_4^-$  with the vertices  $x_i, y_i, u_i, v_i$  where  $u_i, v_i$  are base vertices and  $v_i = u_{i+1}$  for  $i \in \{1, \dots, k-1\}$ . For  $i \in \{1, \dots, k-1\}$ , we lift the edges  $\{x_i, v_i\}$  and  $\{u_{i+1}, x_{i+1}\}$ . Afterward we contract the edges  $\{u_1, x_1\}$ ,  $\{v_1, y_1\}$ ,  $\{v_k, x_k\}$ ,  $\{u_k, y_k\}$ . This gives us the graph  $\Gamma_k$ . Because  $F_k \leq_{\text{lm}} \Gamma_k$ , as shown above, this means that  $F_k \leq_{\text{lm}} L_k$ .  $\square$

**Lemma 3.2.** *Let  $G$  be a 3-connected graph with at least four vertices,  $\{u, v\} \in E(G)$ . Then  $G$  can be contracted to  $K_4$  in such a way that  $\{u, v\}$  is an edge of the obtained graph.*

**Proof.** The graph  $G$  has at least three internally vertex disjoint  $(u, v)$ -paths. Hence, there are at least two vertex disjoint  $(u, v)$ -paths  $P_1, P_2$  that avoid the edge  $\{u, v\}$ . The set  $\{u, v\}$  does not separate  $V(P_1) \setminus \{u, v\}$  and  $V(P_2) \setminus \{u, v\}$ . Therefore, there is a path that joins these sets, and the claim follows.  $\square$

We also need the following proposition.

**Proposition 3.3** ([10]). *There exists a function  $g : \mathbb{N} \rightarrow \mathbb{N}$  such that every graph excluding  $W_k$  and  $K_{3,k}$  as a minor has a tree-decomposition of width at most  $g(k)$  and adhesion at most two.*

Recall that for two vectors of integers  $x = (x_w, \dots, x_1)$  and  $y = (y_w, \dots, y_1)$ ,  $x < y$  lexicographically, if there is  $k \in \{1, \dots, w-1\}$  such that  $x_i = y_i$  for  $i \in \{k+1, \dots, w\}$  and  $x_k < y_k$ . For a tree decomposition  $(\mathcal{X}, T)$  of width  $w$ , denote by  $b_i$  the number of bags of size  $i$  for  $i \in \{1, \dots, w+1\}$ . We say that such a tree decomposition  $(\mathcal{X}, T)$  with adhesion at most two is *minimal*, if the vector  $b = (b_w, \dots, b_1)$  is lexicographically minimal, where the minimum is taken over all tree decompositions of width at most  $w$  and adhesion at most two. We need the following property of minimal tree decompositions.

**Lemma 3.4.** Let  $\mathcal{X} = \{X_i \mid i \in V(T)\}$  be a minimal tree decomposition of a connected graph  $G$  of minimum degree at least 3. For a bag  $X_i$  denote by  $\overline{G}[X_i]$  the graph obtained from  $G[X_i]$  by the addition of (non-existing) edges  $\{u, v\}$  for the pairs of vertices  $u, v \in X_i$  such that there is another bag  $X_j$  with  $X_i \cap X_j = \{u, v\}$ . Then the following holds.

- (a) No bag is a subset of another bag.
- (b) For each bag  $X_i$ , either (i)  $G[X_i]$  is a bridge in  $G$  and  $i$  is not a leaf of  $T$ , or (ii)  $\overline{G}[X_i]$  is a triangle and for each  $u \in V(X_i)$ , there is another bag  $X_j$  with  $u \in X_j$ , or (iii)  $\overline{G}[X_i]$  is a 3-connected graph with at least four vertices.
- (c) If  $X_i, X_j$  are distinct bags,  $X_i \cap X_j = \{x, y\}$ , then there is a  $(x, y)$ -path in  $G$  that avoids the vertices of  $X_i \setminus \{x, y\}$ .

**Proof.** Statement (a) follows directly from the minimality. Notice that it implies that there are no bags of size one, since  $G$  is a connected graph of minimum degree at least 3.

We now prove (b). Let  $X_i = \{u, v\}$  be a bag of size two. We claim that  $v$  and  $u$  are adjacent. To see this, assume to the contrary that  $u$  and  $v$  are not adjacent. Let also  $e_v$  and  $e_u$  be the first and the last edge of a path in the connected graph  $G$  starting from  $v$  and finishing at  $u$ . Let also  $X_{i_u}$  (resp.  $X_{i_v}$ ) be a bag where the edge  $e_u$  (resp.  $e_v$ ) is contained. As  $G$  is connected,  $i$  cannot be in the path of  $T$  connecting  $i_v$  and  $i_u$ . Therefore, we may assume that either  $i_v$  is in the path of  $T$  connecting  $i$  and  $i_u$  or that  $i_u$  is in the path of  $T$  connecting  $i$  and  $i_v$ . In both cases, the third condition of the definition of a tree decomposition implies that either  $u \in X_{i_v}$  or that  $v \in X_{i_u}$ , a contradiction to (a). Hence,  $u, v$  are adjacent, and because  $u, v$  are not included in another bag,  $\{u, v\}$  is a bridge. Clearly,  $i$  cannot be a leaf of  $T$ , as  $G$  has no vertices of degree one.

Now suppose that  $X_i = \{u, v, w\}$  be a bag of size 3. Since the minimum degree of  $G$  is at least 3, each vertex of  $X_i$  is included in another bag. We are left to prove that  $\overline{G}[X_i]$  is a triangle. To obtain a contradiction, assume that  $u$  and  $v$  are not adjacent. Then there is no bag  $X_j, j \neq i$ , with  $u, v \in X_j$ . We modify the tree decomposition as follows. The node  $i$  is replaced by two adjacent nodes  $i', i''$ . Let  $X_{i'} = \{u, w\}$  and  $X_{i''} = \{v, w\}$ . For each  $j$  such that  $X_j \cap X_i \neq \emptyset$ , we join  $j$  with  $i'$  by an edge if  $X_i \cap X_j \subseteq \{u, w\}$ , and we join  $j$  with  $i''$  if  $X_i \cap X_j = \{v\}$  or  $X_i \cap X_j = \{v, w\}$ . We obtain a tree decomposition, where a bag of size three is replaced by two bags of size 2. This contradicts the minimality of the original tree decomposition.

Finally suppose that  $X_i = \{u_1, \dots, u_p\}$  is a bag of size  $p \geq 4$ . To obtain a contradiction, assume that  $H = \overline{G}[X_i]$  is not 3-connected. Then it has a cut set  $S$  of size at most two. Let  $X$  be the set of vertices of a component of the graph obtained from  $H$  by the removal of  $S$ . Let  $Y = X \cup S$  and  $Z = V(H) \setminus X$ . Notice that for any bag  $X_j, X_i \cap X_j \subseteq Y$  or  $X_i \cap X_j \subseteq Z$ . We modify the tree decomposition as follows. The node  $i$  is replaced by two adjacent nodes  $i', i''$ . Let  $X_{i'} = Y$  and  $X_{i''} = Z$ . For each  $j$  such that  $X_j \cap X_i \neq \emptyset$ , we join  $j$  with  $i'$  by an edge if  $X_i \cap X_j \subseteq Y$  and  $X_i \cap X_j \cap X \neq \emptyset$ , and we join  $j$  with  $i''$  if  $X_i \cap X_j \subseteq Z$ . We obtain a tree decomposition, where a bag of size  $p$  is replaced by two bags of size at most  $p - 1$ . This contradicts the minimality of the original tree decomposition. Hence we have proven (b).

Now we prove (c). Suppose that  $X_i, X_j$  are distinct bags,  $X_i \cap X_j = \{x, y\}$  and  $x, y$  are not adjacent. To obtain a contradiction, assume that there are no  $(x, y)$ -paths in  $G$  that avoid the vertices of  $X_i \setminus \{x, y\}$ . Let  $T$  be rooted in  $i$ . The root defines the parent–child relation on  $V(T)$ . Clearly,  $j$  is a child of  $i$ . Denote by  $p$  the last descendant of  $i$  with the same property as  $i$ , i.e.,  $p$  has a child  $q, X_p \cap X_q = \{u, v\}$  and there are no  $(u, v)$ -paths in  $G$  that avoid the vertices of  $X_p \setminus \{u, v\}$ , and no child of  $p$  satisfies this condition. Then  $X_q$  has at least three vertices, and the graph obtained from  $\overline{G}[X_q]$  by the removal of the edge  $\{u, v\}$  is disconnected, but it contradicts (b).  $\square$

Now we are in position to prove [Theorem 1.3](#).

**Proof of Theorem 1.3.** We set  $k = 2n(2n - 1)$  and assume that  $G$  does not contain  $F_k$  as a lift-minor. Also, keep in mind that  $k > 2$ . From [Lemmas 2.1](#) and [2.3](#) it is enough to prove that  $|V(G)|$  cannot be bigger than  $f(k)$  where  $f$  is a function that will be determined later in the proof.

Notice that by [Lemma 3.1](#),  $W_k$  and  $K_{3,k}$  both contain  $F_k$  as a lift-minor. Hence, by [Proposition 3.3](#),  $G$  has a tree-decomposition of width at most  $g(k)$  and adhesion at most two. We assume that  $\mathcal{X} = \{X_i \mid i \in V(T)\}$  is a minimal and, subject to this condition, for a given  $\mathcal{X}$ , a tree  $T$  with the maximum number of leaves is chosen.



Let  $\mathcal{L} \subseteq \mathcal{X}$  be the set of bags corresponding to the leaves of  $T$ . Our strategy is to observe that if the size of  $G$  is big enough, then either  $T$  has many leaves or there is a long path in  $T$  with all vertices of degree two in  $T$ . Then we construct a lift-minor  $F_k$  using either the leaf-bags or the path-bags. For this, we first bound the number of leaves in  $T$ . Then we take the size of  $G$  to be sufficiently big so that, given that both the treewidth of  $\mathcal{X}$  and the number of leaves in  $T$  are bounded, we can force the existence of a path in  $T$ . We need the following claim.

**Claim 1.** *There is a function  $f_1$  such that if  $F_k \not\leq_{\text{lm}} G$ , then  $|\mathcal{L}| < f_1(k)$ .*

**Proof of Claim 1.** Let us assume that  $|\mathcal{L}| \geq f_1(k)$  for some  $f_1$  that will be determined at the end of the proof of this claim and consider the graphs  $L_X = G[X]$ , for each  $X \in \mathcal{L}$ . There are at most two vertices in each  $L_X$  that have neighbors outside  $X$  in  $G$ . Let  $S_X$  be the set of such vertices for each  $X \in \mathcal{L}$ . Denote by  $\bar{L}_X$  the graph obtained from  $L_X$  by joining vertices of  $S_X$  by an edge (if  $|S_X| = 1$ , then  $\bar{L}_X = L_X$ ). By Lemma 3.4,  $X$  has at least four vertices and  $L_X$  is 3-connected for each  $X \in \mathcal{L}$ . We set

$$\mathcal{S}_1 = \{S_X \mid X \in \mathcal{L} \text{ and } |S_X| = 1\} \quad \text{and} \quad \mathcal{S}_2 = \{S_X \mid X \in \mathcal{L} \text{ and } |S_X| = 2\}.$$

If  $S_X \in \mathcal{S}_1$ , then the 3-connectivity of  $L_X$  implies that  $K_4$  is a contraction of  $\bar{L}_X = L_X$  by Tutte's theorem (see c.f. [5]). In case  $|S_X| = 2$ , we define  $L_X^-$  as the graph taken from  $L_X$  by removing, if it exists, the edge with endpoints in  $S_X$ . The 3-connectivity of  $\bar{L}_X$  and Lemma 3.2 imply that  $L_X^-$  can be contracted to  $K_4^-$  in a way that the two base vertices are the vertices of  $S_X$ .  $\square$

We now construct an auxiliary graph  $J$  by taking  $G^- = G[\bigcup_{X \in \mathcal{X} \setminus \mathcal{L}} X]$  and then, for every  $S \in \mathcal{S}_2$ , adding an edge connecting the two vertices of  $S$  (if such an edge already exists, then do not add it). Let us call *essential* the edges connecting in  $J$  the two vertices of some  $S \in \mathcal{S}_2$  (notice that an essential edge of  $J$  is not necessarily present in  $G$ ).

We assign weights to the vertices and edges of  $J$ : each vertex  $v \in V(J)$  receives weight  $|\{L \in \mathcal{L} \mid \{v\} = S_L\}|$  and each edge  $e \in E(J)$  receives weight  $|\{L \in \mathcal{L} \mid e = S_L\}|$ . Observe that the essential edges are *exactly* those with positive weights and recall that the sum of the weights of the edges and vertices of  $J$  is at least  $f_1(k)$ . We prove a series of subclaims.

**Subclaim 1.1.** *The sum of the weights of the vertices in  $J$  is less than  $k$ .*

**Proof of Subclaim 1.1.** Suppose that is not correct. Then contract in  $G$  all edges that do not belong to some of the graphs in  $\{L_X \mid S_X \in \mathcal{S}_1\}$  and obtain a graph that, in turn, can be contracted to  $M_k$ . But then, from Lemma 3.1,  $G$  should contain  $F_k$  as a lift-minor, a contradiction.  $\square$

**Subclaim 1.2.** *There is a function  $f_2$  such that  $J$  does not have more than  $f_2(k)$  blocks with essential edges.*

**Proof of Subclaim 1.2.** Notice that for each block  $B$  of  $J$ , there is a unique block  $B'$  of  $G$  such that  $V(B) \subseteq V(B')$ , and for different blocks  $B_1, B_2$ ,  $V(B_1)$  and  $V(B_2)$  are included in distinct blocks of  $G$ . Observe also that if  $B$  is a block of  $J$  with at least one essential edge, then the corresponding block  $B'$  in  $G$  can be contracted to  $K_4$  by Lemma 3.4. That way, we have that  $G$  can be contracted to a bridgeless graph  $W$  where each of its blocks is a  $K_4$  and such that the number of blocks in  $W$  is equal to the number of blocks in  $J$  with essential edges. Notice that  $W$  cannot contain a cut vertex  $w$  with the property that  $W - w$  has  $k$  or more connected components, otherwise  $W$  could be contracted to  $M_k$  and therefore  $F_k \leq_{\text{lm}} M_k \leq_{\text{lm}} G$  by Lemma 3.1; a contradiction. Moreover, the diameter of  $W$  should be less than  $k$ , otherwise  $W$  contains  $L_k$  as a minor. Then  $F_k \leq_{\text{lm}} L_k \leq_{\text{lm}} G$  by Lemma 3.1, a contradiction. It is now easy to verify that the number of blocks in  $W$  is bounded by some function  $f_2$  of  $k$ . The subclaim follows.  $\square$

From Subclaim 1.1, the sum of weights of the edges of  $J$  is more than  $f_1(k) - k$ . From Subclaim 1.2, one of the blocks of  $J$  denoted by  $B$  should have total-edge weight at least  $\frac{f_1(k) - k}{f_2(k)}$ .

We now construct the graph  $B^*$  from  $B$  by repeatedly removing or contracting non-essential edges: for a non-essential edge  $\{u, v\}$ , if  $\{u, v\}$  is a cut set in the already constructed graph, then we remove the edge, else we contract the edge. Notice that a non-essential edge can be identified with an essential



one after one of these operations (in such a case, such a new edge is essential). Observe also that these operations maintain 2-connectivity. Hence,  $B^*$  is 2-connected. If during such a contraction two edges become one, the weight of the new edge is the sum of the weights of the two edges. Notice that the total edge-weight of  $B^*$  is the same as in  $B$ , that is at least  $\frac{f_1(k)-k}{f_2(k)}$ . Notice also that at most two edges of zero weight may survive in  $B^*$  and this may happen only when  $B^*$  is a triangle where one or two of its edges have positive weights. Clearly, none of the edges in  $B^*$  may have a weight of at least  $k$  as, then, the same sequence of edge contractions and removals in  $G$  would create a graph that contains  $N_k$  as a minor. Then  $F_k \leq_{\text{lm}} N_k \leq_{\text{lm}} G$  by Lemma 3.1; a contradiction. We obtain that the total weight of the edges in  $B^*$  is lower bounded by  $\frac{f_1(k)-k}{k \cdot f_2(k)}$ . In what follows, we will take  $f$  to be big enough so that this lower bound is greater than 2, and therefore we may assume that all edges of  $B^*$  have positive weight. This implies that

$$|E(B^*)| \geq \frac{f_1(k) - k}{k \cdot f_2(k)}. \quad (1)$$

Our next step is to observe that the maximum degree of  $B^*$  is less than  $k$ . Suppose towards a contradiction that some vertex  $y$  of  $B^*$  is incident with at least  $k$  edges. Recall that  $B^*$  is 2-connected and thus  $B^* - y$  is connected. Therefore, if we contract in  $B^*$  all edges that are not incident to  $y$ , we create a single edge with total weight at least  $k$ . As before, this implies that  $F_k \leq_{\text{lm}} N_k \leq_{\text{lm}} G$ ; a contradiction.

Our next observation is that every path in  $B^*$  has length at most  $k - 1$ . Indeed, a path of length at least  $k$  in  $B^*$  would imply the existence in  $G$  of  $L_k$  as a minor, a contradiction, since by Lemma 3.1  $F_k \leq_{\text{lm}} L_k \leq_{\text{lm}} G$ .

According to the two observations above,  $B^*$  has at most  $f_3(k)$  edges for some function  $f_3$ . This, combined with (1), implies that  $f_3(k) \geq \frac{f_1(k)-k}{k \cdot f_2(k)}$  for some specific choice of the functions  $f_2$  and  $f_3$ . If we now take  $f_1$  to be big enough so that this inequality is violated, we have a contradiction and the claim follows.

Notice that the fact that each bag of  $\mathcal{X}$  has at most  $g(n(n-1))$  vertices implies that  $\mathcal{X}$  has at least  $f(n)/g(k)$  bags. Therefore, the tree  $T$  has  $\geq f(n)/g(k)$  vertices and from the above claim, less than  $f_1(k)$  of them are leaves (recall that  $k = n(n-1)$ ). But then we can choose the function  $f$  such that  $T$  contains a path  $P$  of  $24(k+1)^3 + 3$  vertices such that all internal vertices of  $P$  have degree two in  $T$ . By the fact that the minimum degree of  $G$  is at least 3, we obtain that at most half of the graphs induced by the bags corresponding to the vertices of  $P$  are bridges. We call the bags corresponding to these bridges of  $G$  *bridge edges* of  $G$  and we may assume that this path  $P$  has at least  $12(k+1)^3$  internal vertices that correspond to bags that are not inducing bridges in  $G$ . Let  $H$  be the graph obtained from  $G[\bigcup_{i \in V(P)} X_i]$  by contracting all bridge edges. Our aim is to arrive at a contradiction by showing that  $H$  (and therefore  $G$  as well) contains either  $L_k$ , or  $F_k$ , or  $\Gamma_k$  as a minor. Notice that  $\mathcal{X}$  gives rise to a path decomposition  $\mathcal{X}' = \{X_0, \dots, X_{r+1}\}$  of  $H$  containing at least  $12(k+1)^3 + 2$  bags (we first crop from  $\mathcal{X}$  the bags corresponding to  $P$  and then we suppress bridge bags). Recall that the number of leaves in  $T$  is maximum. Then each bag  $X_i$  of  $\mathcal{X}'$  can be of one of the following types:

- (1-1-type)  $Q_i = H[X_i]$  is a 3-connected graph. Moreover, if  $i \in \{1, \dots, r\}$  then such a  $X_i$  contains two vertices  $x_l^i$  and  $x_r^i$  such that  $\{x_l^i\} = X_i \cap X_{i-1}$  and  $\{x_r^i\} = X_i \cap X_{i+1}$ .
- (1-2-type)  $Q_i = H[X_i]$  contains three vertices  $x_l^i$ ,  $x_{ru}^i$  and  $x_{rd}^i$  such that the addition in  $H[X_i]$  of the edge  $\{x_{ru}^i, x_{rd}^i\}$  makes it 3-connected or a triangle (we denote this enhanced graph by  $\bar{Q}_i$ ). Moreover, if  $i \in \{1, \dots, r\}$ , then  $\{x_l^i\} = X_i \cap X_{i-1}$ ,  $\{x_{ru}^i, x_{rd}^i\} = X_i \cap X_{i+1}$  and  $x_l^i \notin \{x_{ru}^i, x_{rd}^i\}$ .
- (2-1-type)  $Q_i = H[X_i]$  contains three vertices  $x_{lu}^i$ ,  $x_{ld}^i$  and  $x_r^i$  such that the addition in  $H[X_i]$  of the edge  $\{x_{lu}^i, x_{ld}^i\}$  makes it 3-connected or a triangle (we denote this enhanced graph by  $\bar{Q}_i$ ). Moreover, if  $i \in \{1, \dots, r\}$ , then  $\{x_{lu}^i, x_{ld}^i\} = X_i \cap X_{i-1}$ ,  $\{x_r^i\} = X_i \cap X_{i+1}$  and  $x_r^i \notin \{x_{lu}^i, x_{ld}^i\}$ .
- (2-2-type)  $Q_i = H[X_i]$  contains the vertices  $x_{lu}^i$ ,  $x_{ld}^i$ ,  $x_{ru}^i$  and  $x_{rd}^i$  where  $x_{lu}^i \neq x_{ld}^i$ ,  $x_{ru}^i \neq x_{rd}^i$ ,  $|\{x_{lu}^i, x_{ld}^i, x_{ru}^i, x_{rd}^i\}| \in \{3, 4\}$ , and the addition in  $H[X_i]$  of the edges  $\{x_{ru}^i, x_{rd}^i\}$  and  $\{x_{lu}^i, x_{ld}^i\}$  makes it 3-connected (we denote this enhanced graph by  $\bar{Q}_i$ ). Moreover, if  $i \in \{1, \dots, r\}$ , then  $\{x_{lu}^i, x_{ld}^i\} = X_i \cap X_{i-1}$  and  $\{x_{ru}^i, x_{rd}^i\} = X_i \cap X_{i+1}$ .

Notice that for each  $i \in \{0, \dots, r\}$ , if  $X_i$  is of  $x$ - $y$ -type and  $X_{i+1}$  is of  $x'$ - $y'$ -type, then  $y = x'$  and that  $y$  is the cardinality of the set  $S_i = X_i \cap X_{i+1}$ . Observe also that for any  $i \in \{1, \dots, r-1\}$ ,  $S_{i-1} \setminus S_i \neq \emptyset$  and  $S_i \setminus S_{i-1} \neq \emptyset$ , since otherwise if  $S_{i-1} \subseteq S_i$  or  $S_i \subseteq S_{i-1}$ , then the node  $i$  of  $T$  could be made a leaf, but this contradicts the choice of  $T$ . Notice that each  $\bar{Q}_i$  is either a triangle or 3-connected by Lemma 3.4. For  $1 \leq i \leq j \leq r$ , we let  $H_{ij} = H[\bigcup_{h \in \{i, \dots, j\}} X_h]$ . We need some properties of  $H_{ij}$  given in the next four claims.

**Claim 2.** Suppose that  $|S_{i-1}| = 1$ ,  $|S_j| = 1$ , and for  $h \in \{i, \dots, j-1\}$ ,  $|S_h| = 2$ . Then the graph  $H_{ij}$  contains three paths  $P_1, P_2, P^*$ , where  $P_1, P_2$  are internally vertex disjoint  $(x_i^j, x_r^j)$ -paths,  $P^*$  joins an internal vertex of  $P_1$  with some internal vertex of  $P_2$  and avoids  $x_i^j, x_r^j$ .

**Proof of Claim 2.** The graph  $H_{ij}$  is 2-connected. Hence, there are two internally vertex disjoint  $(x_i^j, x_r^j)$ -paths  $P_1$  and  $P_2$ . Notice that for each  $h \in \{i, \dots, j-1\}$ , each of the paths  $P_1$  and  $P_2$  contains exactly one vertex of  $S_h$ .

If  $j = i$ , then  $H$  is 3-connected, and because the minimum degree of  $G$  is at least 3,  $H_{ij}$  has at least four vertices. Moreover, as  $H$  is 3-connected we may assume that  $P_1, P_2$  have internal vertices. Observe also that  $\{x_i^j, x_r^j\}$  is not a cut set of  $H$ . Hence, there is a path  $P^*$  that joins an internal vertex of  $P_1$  with some internal vertex of  $P_2$ .

Suppose that  $j > i$ . Notice that  $P_1, P_2$  have internal vertices because  $x_i^j \notin \{x_{ru}^i, x_{rd}^i\}$  and  $x_r^j \notin \{x_{lu}^j, x_{ld}^j\}$ . If for every  $h \in \{i, \dots, j\}$ ,  $|X_h| = 3$ , then there is  $h \in \{i, \dots, j-1\}$  such that  $x_{lu}^h, x_{ld}^h$  are adjacent in  $G$ , since  $G$  has no vertices of degree two. Then such an edge forms a path between an inner vertex of  $P_1$  and an inner vertex of  $P_2$ . Assume that there is an  $h \in \{i, \dots, j\}$  such that  $|X_h| \geq 4$ . By Lemma 3.4,  $\bar{Q}_h$  is 3-connected. If  $h = i$ , then  $x_{ru}^h, x_{rd}^h$  are joined in  $\bar{Q}_h$  by at least three internally vertex disjoint paths. At least one of these paths avoids  $x_r^h$  and the edge  $\{x_{ru}^h, x_{rd}^h\}$  and we have  $P^*$ . If  $h = j$ , then we find  $P^*$  by the same arguments using the symmetry. Let  $i < h < j$ . Then  $x_{ru}^h, x_{rd}^h$  are joined in  $\bar{Q}_h$  by at least three internally vertex disjoint paths, and at least one of these paths avoids the edges  $\{x_{lu}^h, x_{ld}^h\}$  and  $\{x_{ru}^h, x_{rd}^h\}$ .  $\square$

**Claim 3.** Suppose that  $i < r$ ,  $|S_{i-1}| = |S_i| = |S_{i+1}| = 2$  and  $S_{h-1} \cap S_h \cap S_{h+1} = \{u\}$ . Then the graph  $H_{i+1}$  contains two paths  $P, P^*$ , where  $P$  joins the unique vertices  $S_{i-1} \setminus \{u\}, S_{i+1} \setminus \{u\}$  and avoids  $u$ ,  $P^*$  joins an internal vertex of  $P$  with  $u$  and avoids vertices of  $S_{i-1} \cup S_{i+1} \setminus \{u\}$ .

**Proof of Claim 3.** Assume without loss of generality that  $u = x_{ld}^{i+1} = x_{rd}^{i+1} = x_{rd}^{i+1}$  and  $S_{i-1} \setminus (S_i \cup S_{i+1}) = \{x_{lu}^i\}, S_{i+1} \setminus (S_{i-1} \cup S_i) = \{x_{ru}^{i+1}\}$ .

The graph  $\bar{H}_{i+1}$  obtained from  $H_{i+1}$  by the addition of edges  $\{x_{lu}^i, u\}$  and  $\{x_{ru}^{i+1}, u\}$  is 2-connected. Hence, there is a  $(x_{lu}^i, x_{ru}^{i+1})$ -path  $P$  in  $\bar{H}_{i+1}$  that avoids  $u$ . Clearly,  $P$  is a path in  $H_{i+1}$ . This path contains at least one internal vertex  $x_{ru}^i$ . If  $|X_i| = |X_{i+1}| = 3$ , then  $x_{ru}^i, x_{rd}^i$  are adjacent in  $G$ , since  $x_{ru}^i$  has degree at least 3. Then this edge forms a path between an inner vertex of  $P$  and  $u$ . Assume that  $|X_i| \geq 4$ . By Lemma 3.4,  $\bar{Q}_i$  is 3-connected. Then  $x_{ru}^i, x_{rd}^i$  are joined in  $\bar{Q}_i$  by at least three internally vertex disjoint paths, and at least one of these paths avoids the edges  $\{x_{lu}^i, x_{ld}^i\}$  and  $\{x_{ru}^i, x_{rd}^i\}$ . We take this path as  $P^*$ . If  $|X_{i+1}| \geq 4$ , then we find  $P^*$  by symmetrically applying the same arguments.  $\square$

**Claim 4.** Suppose that for  $h \in \{i-1, \dots, j\}$ ,  $|S_h| = 2$ , and  $S_{i-1} \cap S_i = \emptyset$ . Then the graph  $H_{ij}$  contains two disjoint paths  $P_1, P_2$  joining the vertices in  $\{x_{ld}^i, x_{lu}^i\}$  with the vertices in  $\{x_{rd}^j, x_{ru}^j\}$ .

**Proof of Claim 4.** The graph  $\bar{H}_{ij}$  obtained from  $H_{ij}$  by the addition of edges  $\{x_{ld}^i, x_{lu}^i\}$  and  $\{x_{rd}^j, x_{ru}^j\}$  is 2-connected. If we subdivide the edges  $\{x_{rd}^i, x_{ru}^i\}, \{x_{rd}^j, x_{ru}^j\}$ , and denote the obtained vertices of degree two by  $u$  and  $v$  respectively, then the obtained graph contains two internally vertex disjoint  $(u, v)$ -paths. The claim follows immediately.  $\square$

**Claim 5.** Suppose that for  $t \in \{i+1, \dots, j-1\}$ ,  $S_i \cap S_t = \emptyset, S_t \cap S_j = \emptyset$  and for  $h \in \{i-1, \dots, j\}$ ,  $|S_h| = 2$ . Then the graph  $H_{ij}$  contains paths  $P_1, P_2, P^*$ , where  $P_1, P_2$  are disjoint paths joining the vertices in  $\{x_{ld}^i, x_{lu}^i\}$  with the vertices in  $\{x_{rd}^j, x_{ru}^j\}$ , and  $P^*$  joins a vertex of  $P_1$  with some vertex of  $P_2$ .

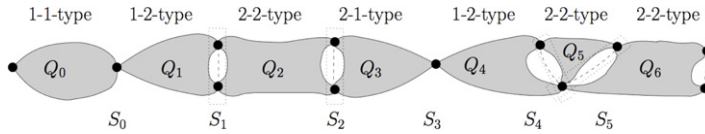


Fig. 4. The types of bags in  $\mathcal{X} = \{X_0, \dots, X_{r+1}\}$ .

**Proof of Claim 5.** The paths  $P_1$  and  $P_2$  exist by Claim 4. Without loss of generality we assume that  $P_1$  is a  $(x_{ld}^i, x_{rd}^i)$ -path and  $P_2$  is a  $(x_{lu}^i, x_{ru}^i)$ -path.

If for every  $h \in \{i, \dots, j\}$ ,  $|X_h| = 3$ , then there is  $h \in \{i, \dots, j\}$  such that  $x_{lu}^h, x_{ld}^h$  are adjacent in  $G$ , since otherwise the vertices of  $S_t$  would have degree two. Then such an edge forms a path between  $P_1$  and  $P_2$ . Assume that there is an  $h \in \{i, \dots, j\}$  such that  $|X_h| \geq 4$ . By Lemma 3.4,  $\overline{Q}_h$  is 3-connected. Then  $x_{lu}^h, x_{rd}^h$  are joined in  $\overline{Q}_h$  by at least three internally vertex disjoint paths, and at least one of these paths avoids the edges  $\{x_{lu}^h, x_{ld}^h\}$  and  $\{x_{ru}^h, x_{rd}^h\}$ . Clearly, this path joins  $P_1$  and  $P_2$  in  $H_{ij}$ .  $\square$

Now we are ready to complete the proof of Theorem 1.3. Consider the sequence  $S_1, \dots, S_r$ . Recall that  $r \geq 12(k+1)^3$ .

First, we show that the sequence  $|S_1|, \dots, |S_r|$  contains at most  $k$  1's. Suppose that  $|S_{h_1}| = \dots = |S_{h_{k+1}}| = 1$  for some  $1 \leq h_1 < \dots < h_{k+1} \leq r$  and that  $|S_h| = 2$  for all  $1 \leq h \leq r$  with  $h \notin \{h_1, \dots, h_{k+1}\}$ . Then we apply Claim 2 for  $H_{h_1 h_2}, \dots, H_{h_k h_{k+1}}$  and conclude that  $H$  contains  $L_k$  as a minor which gives a contradiction because of Lemma 3.1. As a consequence of this, the sequence  $|S_1|, \dots, |S_r|$  contains a subsequence  $|S_i|, \dots, |S_j|$  formed by at least  $12(k+1)^2$  consecutive 2's (in Fig. 4, this holds for  $i = 1$  and  $j = 2$ ). This also means that for all  $h \in \{i+1, \dots, j\}$ ,  $X_h$  is a 2-2-type bag.

Now we prove that the sequence  $S_i, \dots, S_j$  does not contain any subsequence  $S_{i'}, \dots, S_{j'}$  of more than  $2k$  consecutive elements such that  $\bigcap_{h=i'}^{j'} S_h = \{u\}$ . Otherwise, we apply Claim 3 for  $H_{i'+1 i'+2}, H_{i'+3 i'+4}, \dots, H_{i'+2k-1 i'+2k}$ , and it follows that  $F_k$  is a minor of  $H$ ; a contradiction.

We have that the sequence  $S_i, \dots, S_j$  contains a subsequence  $S_{h_1}, \dots, S_{h_{3k}}$  of  $3k$  pairwise disjoint (not necessarily consecutive) elements. We apply Claim 5 for  $H_{h_1 h_3}, H_{h_4 h_6}, \dots, H_{h_{3k-2} h_{3k}}$  and Claim 4 for  $H_{h_3 h_4}, H_{h_6 h_7}, \dots, H_{h_{3k-3} h_{3k-2}}$  and observe that  $\Gamma_k$  is a minor of  $H$ , a contradiction.  $\square$

#### 4. On the structure of lift-contraction-free graphs

Given a graph  $G$  and a subset  $S$  of  $V(G)$ , we denote by  $N_G(S)$  the set of vertices not in  $S$  that are neighbors of vertices in  $S$ . We also define  $\overline{N}_G(S) = N_G(S) \cup S$ . Theorem 1.3 implies the following structural theorem on the graphs excluding some graph  $H$  as a lift-contraction. We call a vertex set  $R$  of a graph  $G$  2-central if for every connected component  $C$  of  $G \setminus R$ , it holds that  $G[\overline{N}_G(V(C))]$  has treewidth at most two and  $|N_G(V(C))| \leq 2$ . We need the following observation.

**Observation 4.1.** Let  $G$  be a graph with a 2-central set  $R$  and let  $G^+$  be the graph obtained from  $G$  by the consecutive application of the following operations: (i) edge subdivisions and (ii) additions of a new vertex adjacent to either a single vertex or two adjacent vertices in the already constructed graph. Then  $R$  is a 2-central set in  $G^+$  as well.

**Theorem 4.2.** There exists a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that every connected graph  $G$  that does not contain an  $h$ -vertex graph  $H$  as a lift-contraction contains a 2-central set  $R$  of at most  $f(h)$  vertices.

**Proof.** Let  $f$  be the function that exists by Theorem 1.3. Assume that there is a minimum size counterexample  $G$  and let  $n = |V(G)|$ . Clearly,  $n > f(h)$  as any graph of at most  $f(h)$  vertices satisfies trivially the property of the theorem. However, from Theorem 1.3, a graph with more than  $f(h)$  vertices that does not contain  $H$  as a lift-contraction, should contain some vertex  $v$  of degree at most two. We contract an edge incident with  $v$  in the connected graph  $G$ , and denote by  $G'$  the obtained graph. Notice that the graph  $G'$  also excludes  $H$  as a lift-contraction and, as  $|V(G')| < n$ ,  $G'$  contains a 2-central set  $R$  of at most  $f(h)$  vertices. From Observation 4.1,  $R$  is also a 2-central set in  $G$ .  $\square$

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## References

- [1] E. Birmelé, J. Bondy, B. Reed, Brambles, prisms and grids, in: A. Bondy, J. Fonlupt, J.-L. Fouquet, J.-C. Fournier, J.L. Ramírez Alfonsín (Eds.), *Graph Theory in Paris*, in: *Trends in Mathematics*, Birkhäuser, Basel, 2007, pp. 37–44.
- [2] B. Bollobás, A. Thomason, Proof of a conjecture of Mader, Erdős, and Hajnal on topological complete subgraphs, *European J. Combin.* 19 (1998) 883–887.
- [3] C. Chun, G. Ding, B. Oporowski, D.L. Vertigan, Unavoidable parallel minors of 4-connected graphs, *J. Graph Theory* 60 (4) (2009) 313–326.
- [4] M. DeVos, Z. Dvořák, J. Fox, J. McDonald, B. Mohar, D. Scheide, Minimum degree condition forcing complete graph immersion, Technical Report, Cornell University, 2010.
- [5] R. Diestel, *Graph Theory*, third ed., in: *Graduate Texts in Mathematics*, vol. 173, Springer-Verlag, 2005.
- [6] M. Grohe, D. Marx, Structure theorem and isomorphism test for graphs with excluded topological subgraphs, in: 44th Symposium on Theory of Computing, STOC 2012, 2012.
- [7] A. Kostochka, Lower bounds on the Hadwiger number of graphs by their average degree, *Combinatorica* 4 (1984) 307–316.
- [8] Lovász, On some connectivity properties of Eulerian graphs, *Acta Math. Acad. Sci. Hungar.* 28 (1976) 129–138.
- [9] J. Matoušek, J. Nešetřil, R. Thomas, On polynomial-time decidability of induced-minor-closed classes, *Comment. Math. Univ. Carolin.* 29 (4) (1988) 703–710.
- [10] B. Oporowski, J. Oxley, R. Thomas, Typical subgraphs of 3- and 4-connected graphs, *J. Combin. Theory Ser. B* 57 (2) (1993) 239–257.
- [11] N. Robertson, P.D. Seymour, Graph minors—a survey, in: *Surveys in combinatorics 1985* (Glasgow, 1985), in: *London Math. Soc. Lecture Note Ser.*, vol. 103, Cambridge University Press, Cambridge, 1985, pp. 153–171.
- [12] R. Thomas, Tree-decompositions of graphs, in: *Lecture notes*, School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332, USA, 1996.
- [13] A. Thomason, An extremal function for contractions of graph, *Math. Proc. Cambridge Philos. Soc.* 95 (1984) 261–265.
- [14] P. Wollan, The structure of graphs not admitting fixed immersion, manuscript.